THE EVOLUTION OF MAXWELL'S EQUATIONS FROM 1862 TO THE PRESENT DAY

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Abstract

Maxwell's Equations have undergone several changes of form in the 150 years since they first emerged. This has been due not only to changes in their mathematical expression and physical interpretation, but also to historical accidents and trends. This article examines: what they were in the beginning; how they evolved into their customary form over the course of 40 years; the significant variations there have been since then; who have been the major proponents of these changes, and finally, what they did, or did not, contribute. Brief explanations of the main mathematical variations are included.

Keywords: Differential forms, Electrodynamics, Electromagnetic engineering education, Geometric algebra, Gibbs, Heaviside, Hertz, History, Lorentz, Maxwell, Maxwell equations, Quaternions.

1 INTRODUCTION

In this article, each equation numbered from (1) to (24) represents a different version of Maxwell's equations! While these equations are familiar to almost every graduate in physics or electrical engineering as

$$\begin{aligned} \nabla\cdot\mathbf{D} &= \rho^{free} \\ \nabla\cdot\mathbf{B} &= 0 \\ \nabla\times\mathbf{E} &= -\partial_t\mathbf{B} \\ \nabla\times\mathbf{H} &= \mathbf{J}^{free} + \partial_t\mathbf{D} \end{aligned} \tag{1}$$

they were not always so clear and concise. In these four partial differential equations, the vector quantities $\mathbf{D}$, $\mathbf{B}$, $\mathbf{E}$ and $\mathbf{H}$ represent the four electromagnetic fields, namely

D, is the electric displacement

B, is the magnetic induction

E, is the electric field intensity, and

H, is the magnetic field intensity.

On the other hand, the sources that generate these fields are the scalar free charge density ρ^{free} and the vector free current density $\mathbf{J}^{free}$. This form and its many variants (apart from trivial changes such as the use of ∂_t, an overdot for $\partial/\partial t$, or even Maxwell's original d/dt) actually owe their existence to several key contributors who tidied up the original equations and polished the underlying physics, and who improved the mathematical tools and notational niceties. Although they are now universally referred to as Maxwell's equations, they do not actually cover all of the equations that Maxwell deemed necessary for the study of electricity and magnetism. As a minimum, the basic constitutive relations

$$\begin{aligned} \mathbf{D} &= \varepsilon_0\mathbf{E} + \mathbf{P} \\ \mathbf{H} &= \frac{1}{\mu_0}\mathbf{B} - \mathbf{M} \end{aligned} \tag{1i}$$

and a force equation,

$$\mathbf{F} = q\left(\mathbf{E} + \mathbf{v}\times\mathbf{B}\right) \tag{1ii}$$

are also required (since these equations may be grouped with Equations (1), we give them the same number with the suffices i , ii …). The constitutive relations bring in the fundamental constants ε_0 and μ_0 that directly relate $\mathbf{D}$ to $\mathbf{E}$ and $\mathbf{H}$ to $\mathbf{B}$ in free space while $\mathbf{P}$ and $\mathbf{M}$ are source densities, respectively the electric and magnetic dipole densities within a material body (and clearly vanish in free space). The force equation, known as the Lorentz force, tells us the electromagnetic force that acts on a point charge q travelling through an electromagnetic field with velocity $\mathbf{v}$. Only $\mathbf{E}$ and $\mathbf{B}$ are involved in this force whereas $\mathbf{D}$ and $\mathbf{H}$ take no direct part.

Our aim in the present article is to sketch out how we got from Maxwell's original equations to the ubiquitous version in Equations (1) by reviewing the roles of the key contributors to its mathematical and physical development. We also go beyond Equations (1), which have now been extant for over 100 years, to address more recent approaches that have found favor in certain areas and which shed further light on the fundamental principles of electrodynamics. For the benefit of the general reader we briefly explain the mathematical formalisms involved in these approaches, but no appreciation of special relativity will be required.

In tracing the story, we attempt to give a realistic idea of how Maxwell's equations appeared at each stage. Despite the historical context, we transfer everything to modern symbols and conventions, including the international system (SI) with which the majority of people are now familiar. While mixing different sets of symbols would have created some confusion, as is well known, switching between one system and another causes various constant factors to appear or disappear at various places (see, for example, [1, Appendix, Table 2]) leading to a potential source of even greater confusion. We can manage better by simply allowing any differences of this sort between SI and the original units to pass unchallenged. In any case, Maxwell did not adhere to any single system that could be found in the table just referred to, and as to symbols, an example will suffice, Initially, Maxwell did not use the symbol **E** for the electric field, nor did he use subscripts. Instead, as shown in Figures 1-3, he used *P*, *Q* and *R* for the components of **E** (later, in a similar vein, Hertz used *X*, *Y* and *Z*). This led to vector equations having to be written 'longhand', *i.e*. three times, one for each component. We therefore keep to E_x, E_y and E_z, *etc*., as this will be consistent and clear to all. In addition, the original texts often refer to current and charge when it is actually their densities that are implied (in fact, $\partial_t \mathbf{D}$ is still referred to as the displacement 'current') and the terms 'electric force' and 'magnetic force' may be used in reference to **D** and **H** respectively, rather than to **E** and **B** (which are the only two fields that appear in the Lorentz force). Particular care is needed in these situations to determine whether it is the terminology or the equation that is at fault [2].

Finally, although we can never be absolutely certain exactly who did what first and where the initial ideas originally came from, it is fairly certain that Heaviside and Lorentz both made significant contributions to the clarification and formation of the equations, while Hamilton, Heaviside (again), and Gibbs clearly contributed to their mathematical expression. It is also clear that Oersted, Ampère, Biot, Savart and Faraday were Maxwell's antecedents in the development of electromagnetic theory, while Boltzmann, Hertz, Kirchhoff, Lorenz and Weber all made roughly contemporaneous contributions to it. However, it is not our purpose here to reflect on the degree to which any particular contributor may have directly or indirectly influenced the formulation of his equations. In a brief sketch of the subject it is not always possible to acknowledge every contribution and, in the end, we have simply attempted to be as clear and as accurate as space and time permits.

$$\left.\begin{aligned} \frac{dG}{dz}-\frac{dH}{dy}&=\mu\alpha, \\ \frac{dH}{dx}-\frac{dF}{dz}&=\mu\beta, \\ \frac{dF}{dy}-\frac{dG}{dx}&=\mu\gamma. \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (55)$$

$$\left.\begin{aligned} P&=\mu\gamma\frac{dy}{dt}-\mu\beta\frac{dz}{dt}+\frac{dF}{dt}-\frac{d\Psi}{dx}, \\ Q&=\mu\alpha\frac{dz}{dt}-\mu\gamma\frac{dx}{dt}+\frac{dG}{dt}-\frac{d\Psi}{dy}, \\ R&=\mu\beta\frac{dx}{dt}-\mu\alpha\frac{dy}{dt}+\frac{dH}{dt}-\frac{d\Psi}{dz}. \end{aligned}\right\} \quad . \; . \; . \quad (77)$$

$$R=-4\pi E^2 h,$$

$$\left.\begin{aligned} p&=\frac{1}{4\pi}\left(\frac{d\gamma}{dy}-\frac{d\beta}{dz}-\frac{1}{E^2}\frac{dP}{dt}\right), \\ q&=\frac{1}{4\pi}\left(\frac{d\alpha}{dz}-\frac{d\gamma}{dx}-\frac{1}{E^2}\frac{dQ}{dt}\right), \\ r&=\frac{1}{4\pi}\left(\frac{d\beta}{dx}-\frac{d\alpha}{dy}-\frac{1}{E^2}\frac{dR}{dt}\right). \end{aligned}\right\} \quad . \; . \quad (112)$$

$$\frac{dp}{dx}+\frac{dq}{dy}+\frac{dr}{dz}+\frac{de}{dt}=0 \quad . \; . \; . \; . \; . \; . \quad (113)$$

$$e=\frac{1}{4\pi E^2}\left(\frac{dP}{dx}+\frac{dQ}{dy}+\frac{dR}{dz}\right) \quad . \quad . \quad (115)$$

Figure 1: Maxwell's Equations as they First Appeared in 1861-2

This was Maxwell's first published attempt at a complete set of electrodynamical equations, including displacement current. Apart from the term E^2 that originates in the unnumbered equation, he used exactly the same symbols in the Dynamical Theory, and comparison with Figure 2 shows that, inconsistencies in minus signs excepted, the equations were more or less equivalent. Note that the unnumbered equation above expresses only the x components of the vector equation $\mathbf{E}=\varepsilon^{-1}\mathbf{D}$ (ignoring the minus sign). While the problem with these original equations was the connexion with the molecular vortex model, they did lead Maxwell to the same conclusions: the existence of electromagnetic waves, and the strong likelihood that light was an electromagnetic wave. Elements of this figure were taken from Maxwell's original article as it appears in digitized form on

http://upload.wikimedia.org/wikipedia/commons/b/b8/On_Physical_Lines_of_Force.pdf

For Electromagnetic Momentum	F	G	H	**(A)**
„ Magnetic Intensity	α	β	γ	**(H)**
„ Electromotive Force	P	Q	R	**(E)**
„ Current due to true conduction	p	q	r	**(J)**
„ Electric Displacement	f	g	h	**(D)**
„ Total Current (including variation of displacement)	p'	q'	r'	$(\mathbf{J}+\partial_t \mathbf{D})$
„ Quantity of free Electricity	e			(ρ)
„ Electric Potential	Ψ			(ϕ)

Between these twenty quantities we have found twenty equations, viz.

Three equations of Magnetic Force	(B)
„ Electric Currents	(C)
„ Electromotive Force	(D)
„ Electric Elasticity	(E)
„ Electric Resistance	(F)
„ Total Currents	(A)
One equation of Free Electricity	(G)
„ Continuity	(H)

Figure 2: Maxwell's Twenty Symbols and List of Equations from the *Dynamical Theory*

Maxwell's equations, and the summary of the symbols he used, as they appear on page 486 of the Dynamical Theory. Note that in partial derivatives he wrote d instead of ∂ and the character resembling ρ or ζ in Equation (F) denotes resistivity. The red mark-up indicates the vector and scalar symbols that we use today. Maxwell called the vector potential 'electromagnetic momentum' because $e\partial_t \mathbf{A}$ represents a force acting on e units of negative charge, and this bears a similarity to the familiar mechanical force $m\partial_t \mathbf{p}$, where **p** is mechanical momentum and *m* is mass. Could it have been this analogy that prompted the problem with the signs of *e* in equations (F) and (C)? (Based on the digitized copy of the original article, available on the Royal Society of London's website [4]).

$$\left.\begin{aligned} p' &= p + \frac{df}{dt}, \\ q' &= q + \frac{dg}{dt}, \\ r' &= r + \frac{dh}{dt}, \end{aligned}\right\} \quad . \quad (A) \qquad \left.\begin{aligned} \mu\alpha &= \frac{dH}{dy} - \frac{dG}{dz}, \\ \mu\beta &= \frac{dF}{dz} - \frac{dH}{dx}, \\ \mu\gamma &= \frac{dG}{dx} - \frac{dF}{dy}. \end{aligned}\right\} \quad . \; . \; . \quad (B)$$

$$\left.\begin{aligned} \frac{d\gamma}{dy} - \frac{d\beta}{dz} &= 4\pi p', \\ \frac{d\alpha}{dz} - \frac{d\gamma}{dx} &= 4\pi q', \\ \frac{d\beta}{dx} - \frac{d\alpha}{dy} &= 4\pi r'. \end{aligned}\right\} \quad . \; . \quad (C) \qquad \left.\begin{aligned} P &= \mu\left(\gamma\frac{dy}{dt} - \beta\frac{dz}{dt}\right) - \frac{dF}{dt} - \frac{d\Psi}{dx}, \\ Q &= \mu\left(\alpha\frac{dz}{dt} - \gamma\frac{dx}{dt}\right) - \frac{dG}{dt} - \frac{d\Psi}{dy}, \\ R &= \mu\left(\beta\frac{dx}{dt} - \alpha\frac{dy}{dt}\right) - \frac{dH}{dt} - \frac{d\Psi}{dz}. \end{aligned}\right\} \quad . \; . \quad (D)$$

$$\left.\begin{aligned} P &= kf, \\ Q &= kg, \\ R &= kh. \end{aligned}\right\} \quad . \; . \quad (E) \qquad \left.\begin{aligned} P &= -\varrho p, \\ Q &= -\varrho q, \\ R &= -\varrho r. \end{aligned}\right\} \quad . \; . \quad (F)$$

$$e + \frac{df}{dx} + \frac{dg}{dy} + \frac{dh}{dz} = 0 \quad . \; . \quad (G) \qquad \frac{de}{dt} + \frac{dp}{dx} + \frac{dq}{dy} + \frac{dr}{dz} = 0 \quad . \; . \; . \quad (H)$$

Figure 3: Maxwell's Twenty Equations from the *Dynamical Theory*

Maxwell's twenty equations as they appear over pages 480-485 of the *Dynamical Theory*. Comparison with the earlier equations of 1861-62 shows that a number of terms have different signs. There were twenty equations because he wrote one equation for each vector component and covered a wide range of electric and magnetic phenomena in an attempt to give them a common foundation. What we now consider to be the key electromagnetic equations are therefore a subset of these. (Based on the digitized copy of the original article, available on the Royal Society of London's website [4]).

2 MAXWELL TO LORENTZ

2.1 James Clerk Maxwell

Arguably, the earliest evidence of Maxwell's equations are those, shown in Figure 1, given by James Clerk Maxwell in a four-part article that he published over the course of March 1861 to February 1862 [3]. Here, using the concept of "molecular vortices", he sought a mechanical analogy for the behavior of electromagnetic media, mainly as an aid to understanding how they mediate the two kinds of electromagnetic force. The resulting equations included the novel proposal of a displacement current, leading him to the key conclusion that, not only would media conforming to this model support transverse electromagnetic waves, but that light could well be such a wave.

In December 1864, however, he presented a further paper to the Royal Society of London. In this new article, which we will refer to as the *Dynamical Theory* [4], the essence of these equations was retained but the mechanical analogy was abandoned and replaced by the more abstract notion, originally due to Faraday, of an electromagnetic field* that pervades all space and physical media alike. In so doing, Maxwell avoided the conceptual difficulties of the alternative action-at-a-distance theory of Weber [5, p. 67] and gave, for the first time, a credible mathematical basis for the universal laws governing electromagnetic phenomena, Figures 2 and 3. He had now established the foundational form of Equations (1)–(1)ii above†.

It should be noted, however, that Maxwell brought in not only the electric and magnetic fields into his equations, but also the scalar and vector and potentials. Although his own major innovation was the inclusion of the displacement current $\partial_t \mathbf{D}$ in Ampère's law, he did not have an exact counterpart for the constitutive equation $\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}$; rather, he offered the linear law $\mathbf{D} = \varepsilon \mathbf{E}$ that was held to characterize dielectrics in the same way that $\mathbf{J} = \sigma \mathbf{E}$ characterizes conductors. Similarly, he used $\mathbf{B} = \mu \mathbf{H}$. Although he discussed electric polarization in dielectrics, he did so in terms that were conceptual rather than definitive and so he did not actually distinguish it from what he called electric displacement. The distinction, of course, is clear enough now; polarization exists only in physical media whereas displacement can exist in free space. Nevertheless, the absence of fine detail may have actually helped Maxwell reach his idea of a displacement current, for he seems to have been thinking along the lines that the ether would behave like a real medium, in which case his concept of displacement would apply equally well to both. At that time, fortunately, there were few theoretical obstacles to dissuade him of his notion.

By 1873 Maxwell had consolidated his foundational work on electromagnetics by publishing a two volume work that we will refer to as the *Treatise* [6]. The main advance in his equations from the *Dynamical Theory* to the *Treatise* was notational rather than physical; in the former he started out with the aforementioned 'longhand' form of vector equations with no reference to vectors *per se*. The mathematical concept of vectors [7, Ch. 1] and their representation in terms of indexed components was not in general use at the time. In SI and present day notation, the initial forms of his key equations corresponding to Equations (1) may be rendered as

---

* "…the *Electromagnetic Field*…that part of space which contains and surrounds bodies in electric and magnetic conditions." [4, p.460].

† Notwithstanding this, the force equation was eventually named separately after H. A. Lorentz.

$$
\begin{array}{lll}
(G)\ldots & \rho^{free} + \partial_x D_x + \partial_y D_y + \partial_z D_z = 0 & (1 \text{ equation}) \\
(B)\ldots & B_j = \partial_k A_l - \partial_l A_k & (j,k,l \text{ cyclic} \Rightarrow 3 \text{ equations}) \\
(D)\ldots & E_j = v_k B_l - v_l B_k - \partial_t A - \partial_j \phi & (j,k,l \text{ cyclic} \Rightarrow 3 \text{ equations}) \\
(A)\ldots & \left.\begin{array}{l} J_l = J_l^{free} + \partial_t D_l \\ \partial_j H_k - \partial_k H_j = J_l \end{array}\right\} & \begin{array}{l} (l = x,y,z \Rightarrow 3 \text{ equations}) \\ (j,k,l \text{ cyclic} \Rightarrow 3 \text{ equations}) \end{array} \\
(C)\ldots & &
\end{array}
\qquad (2)
$$

Beside each of these equations is the lettered label that Maxwell's gave them but, for ease of comparison, they have been put in the same order as we find them in Equations (1), with (*A*) and (*C*) being grouped as one. Figure 2, shows Maxwell's symbols and how he enumerated and described his equations, while Figure 3 shows how all 20 of his equations actually appeared. Note that there was an inconsistency in his treatment of charge, so that the sign of ρ^{free} in equation (*G*) is wrong. It was in the two homogeneous equations, (B) and (D), that he employed the vector and scalar potentials **A** and ϕ. Although this may seems a bit odd today, as Heaviside was to show, his equation (*B*) is the same as $\mathbf{B} = \nabla \times \mathbf{A}$, which of course leads to $\nabla \cdot \mathbf{B} = 0$, and similarly (*D*) leads to $\nabla \times \mathbf{E} = -\partial_t \mathbf{B}$ provided that all the v_j are set to zero. Equations (2) are therefore just a different way of expressing Equations (1). However, since the v_j are the components of a velocity, their presence in the original equation also embodies an equivalent of the Lorentz force through a velocity dependent modification of the electric field itself – an idea that is closer indeed to tenets of modern relativity. That aside, he also gave separate equations (*L*)* and (*J*) that, when taken together in the form $\mathbf{F} = q\mathbf{E} + \mathbf{I}dl \times \mathbf{B}$, are also equivalent to the Lorentz force, with $\mathbf{I}dl$ being the same infinitesimal current element that we find in the Biot and Savart force law.

Although Maxwell generally adhered to the use of 'longhand' equations in his treatise, he did make a radical step forward by also introducing W.R. Hamilton's quaternions [7, 8], perhaps as a result of the influence of his close friend P. G. Tait, who had published a treatise on them in 1867 [9]. Since that was just after the *Dynamical Theory*, it gave Maxwell plenty of time to take the new ideas on board and take a step forward in a mathematical direction.

Quaternions have four components, one of which is a scalar and the other three of which are components of a vector; they also embody the needs of complex arithmetic, dot and cross products, and provide a familiar form of notation for the 3D basis vectors: **i**, **j**, **k**. For example, they now made possible the representation of the electric field as the single entity **E**, meaning $E_x\mathbf{i} + E_y\mathbf{j} + E_z\mathbf{k}$, and the creation of vector derivative $\nabla = \partial_x\mathbf{i} + \partial_y\mathbf{j} + \partial_z\mathbf{k}$. An expression such as $\nabla \cdot \mathbf{E}$ would then be written as $-S.\nabla\mathbf{E}$, in which the role of the operator

* The equation label "(*D*)" on p. 491 of Maxwell's original text is presumably a misprint for "(*L*)", which would logically be the next label in sequence.

$S.$ is to pick out the scalar part of the argument, whereas $V.$ picks out the vector part, giving us $\nabla\times\mathbf{E}\equiv V.\nabla\mathbf{E}$. The meaning of $\nabla\mathbf{E}$ is that $\partial_x\mathbf{i}+\partial_y\mathbf{j}+\partial_z\mathbf{k}$ and $E_x\mathbf{i}+E_y\mathbf{j}+E_z\mathbf{k}$ are to be multiplied directly term by term, observing left to right order and using Hamilton's basic rule*, $\mathbf{i}^2=\mathbf{j}^2=\mathbf{k}^2=\mathbf{ijk}=-1$. It is readily verified that $\nabla\mathbf{E}$ has the requisite form of a scalar plus a vector, and leads to $\nabla\mathbf{E}=S.\nabla\mathbf{E}+V.\nabla\mathbf{E}\equiv-\nabla\cdot\mathbf{E}+\nabla\times\mathbf{E}$. Keeping the labeling of his *original* equations for ease of comparison, the equations that we are chiefly interested kept the same meaning but emerged in a new form [6, Arts. 609-19]

$$
\begin{array}{lll}
(G)\ldots & \rho^{free}=-S.\nabla\mathbf{D} & \rho^{free}=\nabla\cdot\mathbf{D} \\
(B)\ldots & \mathbf{B}=V.\nabla\mathbf{A} & \mathbf{B}=\nabla\times\mathbf{A} \\
(D)\ldots & \mathbf{E}=V.\mathbf{vB}-\dot{\mathbf{A}}-\nabla\phi & \mathbf{E}=\mathbf{v}\times\mathbf{B}-\partial_t\mathbf{A}-\nabla\phi \\
(A)\ldots & \left.\begin{array}{l}\mathbf{J}=\mathbf{J}^{free}+\dot{\mathbf{D}} \\ V.\nabla\mathbf{H}=\mathbf{J}\end{array}\right\} & \left.\begin{array}{l}\mathbf{J}=\mathbf{J}^{free}+\partial_t\mathbf{D} \\ \nabla\times\mathbf{H}=\mathbf{J}\end{array}\right\} \\
(C)\ldots & &
\end{array}
\qquad (3)
$$

They are shown as quaternions on the left, whereas on the right they are shown in the usual notation. Although Maxwell chose German letters as symbols for the vectors, as shown in Figure 4, most of them stay with us, albeit in more familiar Roman form, namely **A**, **B**, **D**, **E**, and **H**. In addition to these, Maxwell also introduced the magnetic constitutive equation now written as $\mathbf{B}=\mu_0(\mathbf{H}+\mathbf{M})$, but it is curious that this did not draw him towards a similar electrical version as he still adhered to $\mathbf{D}=\varepsilon\mathbf{E}$. As already explained above, equation (*D*) leads directly to the Lorentz force on a unit point charge. However, Maxwell made a mistake in converting it to a force acting on current and charge densities [6, Art. 618]. In addition to the original error pointed out by Fitzgerald and referred to in Thomson's footnote, it is now also clear that the force acting on a current density should not include the entire displacement current, which is $\partial_t(\varepsilon_0\mathbf{E}+\mathbf{P})$, since only the polarization current $\partial_t\mathbf{P}$ represents an actual current density†. On the other hand, he did fix the sign error in equation (*G*)!

We find that while little else had changed in any physical sense from the equations in his *Dynamical Theory*, and despite Equations (*B*) and (*D*) still not being directly comparable with their present-day forms, it is much easier to see the connexion between these quaternion equations and their vector counterparts. Eventually, however, the quaternion formalism lost

* He is said to have inscribed this equation on a bridge over the river Liffe in Dublin. There were to be no scruples about 'adding apples and pears', or rather scalars and vectors, a practice later banned by the vector analysts.

† This error arose in the *Dynamical Theory* because Maxwell's concept of displacement was based on the movement of charges, even in the ether. Maxwell made other errors; notably, he misinterpreted the force acting on a current as acting on the *conductor* that bears the current [5, Art 501]. This so rankled with one student of the *Treatise*, E. B. Hall, that it set him on the path of discovering the Hall effect [11].

ground to vector analysis, possibly because the latter was seen by many as being much simpler in concept and quite adequate for the purposes of most physicists and engineers.

	Symbol of Vector.	Constituents.
The radius vector of a point..................	ρ	$x \quad y \quad z$
The electromagnetic momentum at a point	$\mathfrak{A}$	$F \quad G \quad H$
The magnetic induction	$\mathfrak{B}$	$a \quad b \quad c$
The (total) electric current	$\mathfrak{C}$	$u \quad v \quad w$
The electric displacement....................	$\mathfrak{D}$	$f \quad g \quad h$
The electromotive force	$\mathfrak{E}$	$P \quad Q \quad R$
The mechanical force	$\mathfrak{F}$	$X \quad Y \quad Z$
The velocity of a point.......................	$\mathfrak{G}$ or $\dot{\rho}$	$\dot{x} \quad \dot{y} \quad \dot{z}$
The magnetic force	$\mathfrak{H}$	$\alpha \quad \beta \quad \gamma$
The intensity of magnetization	$\mathfrak{I}$	$A \quad B \quad C$
The current of conduction	$\mathfrak{K}$	$p \quad q \quad r$

Figure 4: Maxwell's Old German Symbols for the Vectors used in the *Treatise*.

The Old German letters $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{D}$, $\mathfrak{E}$ and $\mathfrak{H}$ correspond to the roman letters **A**, **B**, **D**, **E** and **H**, all of which are the symbols still in regular use today for key electromagnetic quantities. The 'constituents' are the vector components, as they were in the *Dynamical Theory* and [3]. (Reproduced from Art. 618, Vol. 2 of the 1st Edition (1873), of Maxwell's treatise, digitized by Google Books).

2.2 Oliver Heaviside

Oliver Heaviside, who was largely self-taught, was so eager to get to grips with Maxwell's electromagnetic theory that he studied the *Treatise* avidly until he was well enough versed in it to forge his own way ahead with the theory. At about the same time that Maxwell was publishing his treatise, Heaviside was publishing in journals such as the *Electrician* and the *Telegraphic Journal*. This developed into a considerable body of work that he later republished as *Electrical Papers* [10] and *Electromagnetic Theory* [11]. The publication of one such article [10, Vol. 1, Art. 30, §§1-9] commenced in January 1885 and over the course of nine separate issues ("sections"), Heaviside gave the reader his 'rough sketch' of Maxwell's theory.

While Heaviside was an enthusiastic proponent of Maxwell's theory, he was very much less sympathetic to his use of quaternions. He felt that these were difficult and, as a consequence, unpopular, even citing Tait's book as one of the main reasons for this! In his view, quaternions were a handicap that had prevented Maxwell from doing himself full justice and as a result he had not achieved the level of acclaim that he undoubtedly deserved [10, Vol. 1. Art 26, §15]:

> "…there is no question as to the difficulty and the practical inconvenience of the quaternionic system."

Heaviside's main purpose in this serialized article was therefore to give Maxwell's theory a clearer exposure by employing a cut-down *vector algebra* drawn out of the quaternions. Like many of Heaviside's mathematical ideas, this was innovative, but he did not throw the baby out with the bathwater, rather he simply dispensed with the difficulties. He retained vectors and scalars but did not allow addition between them; the square of a vector was to be positive rather than negative, and the multiplication of vectors was to be allowed only through scalar (inner) and vector (cross) products, both of which had simple geometrical meanings. These ideas were introduced in 1882 [10, Vol. 1, XXIV, §1], developed in 1885 [10, vol. 2, Art. 31], and later published in 1891-92 as *Elements of Vectorial Algebra and Analysis* [11, Ch. 3].

This first innovation greatly improved the readability of Maxwell's equations, but as to their actual content, he did not condense 20 into just four, as some authors have claimed, since Maxwell had already achieved effectively the same thing in Equations (3); nor did he even put the four main equations together into a single group as we know them now, but what he produced did contain four familiar looking equations,

$$\begin{aligned}
&(\text{IX})\ldots \quad \operatorname{div}\mathbf{D} = \rho \\
&(\text{V})\ldots \quad \operatorname{div}\mathbf{B} = 0 \\
&(\text{IV})\ldots \quad \operatorname{curl}\mathbf{E} = -\partial_t \mathbf{B} \\
&(\text{III})\ldots \quad \operatorname{curl}\mathbf{H} = \sigma\mathbf{E} + \partial_t \mathbf{D}
\end{aligned} \tag{4}$$

The Roman numerals used here are not actual equation numbers, rather they indicate the installment that each equation appeared in, making it quite clear that they emerged separately over a period of months, along with $\mathbf{D} = \varepsilon\mathbf{E}$, $\mathbf{J}^{cond} = \sigma\mathbf{E}$ and the like, each of his equations more or less matching one from Maxwell's *Treatise*. In these, his second major step forward was to eliminate the potentials in Maxwell's equations (3) by taking the divergence of equation (*B*) and the curl of equation (*D*), from which he also abstracted the $V\mathbf{.vB}$ term to correspond with measuring **E** in a fixed frame of reference rather than on a moving conductor (see the discussion in §2.1). Although these equations were to be repeated frequently throughout Heaviside's vast outpouring of work, the only identifiable 'special set' seems to be (III) and (IV) which, because of their cross-coupling, he called the "duplex equations".

2.3 J. Willard Gibbs

The American scientist J. Willard Gibbs, who gave his name to the Gibbs' phenomenon, is mainly renowned for his work on thermodynamics, a subject in which Maxwell was also much involved. The two men corresponded, but it was generally on this subject rather than electromagnetic theory. Gibbs made other significant advances in mathematics, in particular vector analysis, but all the same he did not contribute directly to the reformulation of

Maxwell's equations; he is, however, credited with providing the familiar vector formalism that we normally use for them today. More or less at the same time that Heaviside was writing on, and using to advantage, his own synthesis of vector analysis, Gibbs produced a pamphlet [13] on very similar ideas as an accompaniment to the lectures he gave at Yale University during 1881-84. A more complete compilation of Gibbs' lectures was published in 1901 by E. B. Wilson as *Vector Analysis* [14]. In his preface, Wilson acknowledges the contribution made by Heaviside:

> "By far the greater part of the material… has been taken from the course of lectures on Vector Analysis delivered…by Professor Gibbs. Some use, however, has been made of the chapters on Vector Analysis in Mr. Oliver Heaviside's *Electromagnetic Theory* … and in Professor Föppl's lectures on *Maxwell's Theory of Electricity* …"

While Wilson briefly included the "duplex equations" in his book (in both differential and integral form) Gibbs' original pamphlet had been purely mathematical. In fact, it had become known to Heaviside only in 1888 [10, Vol. 2, Art. 52, §6, footnote to p. 529]. He later said of Gibbs' likeminded approach [10, Vol. 1, Art. 26, §17, footnote on pp. 271-272]:

> "Professor Willard Gibbs, the author of a valuable work on vector analysis, also ignores the quaternion, abolishes the *minus* sign and the double signification of a vector, following Grassmann rather than Hamilton."

Independently, it would seem, they had the same idea of a vector analysis founded on the simplification of quaternion theory by salvaging the vector core from the rest. Actually, there were strong feelings both for and against quaternions. When P. G. Tait claimed in the preface of his book [9] that "…Gibbs must be ranked as one of the retarders of Quaternion progress…", Heaviside came to Gibbs' defense with "This may be very true; but Professor Gibbs is anything but a retarder of progress in *vector analysis*…" [10, Vol. 2, Art. 52, §6, footnote to p. 529]. The differences between Gibbs' and Heaviside's ideas were minor, for example, Heaviside wrote out *div* and *curl* and used the notation **ab** and V**ab** for the scalar and vector products respectively, whereas Gibbs introduced the notation we use today, $\nabla\cdot$, $\nabla\times$, $\mathbf{a}\cdot\mathbf{b}$ and $\mathbf{a}\times\mathbf{b}$. However, in spite of their overall similarity, it is undeniable that Gibbs' book was a major factor in the dissemination of the language of vector analysis, a language that has prevailed across a wide body of physics and engineering ever since.

2.4 Heinrich Hertz

In mainland Europe, Maxwell's equations were often referred to as the Maxwell-Hertz equations. In his revolutionary 1905 paper on special relativity [15], Albert Einstein not only refers to them as such, but quotes Hertz's 'longhand' formulation in preference to those of either Heaviside, Lorentz or Föppl, who had all adopted vector analysis.

Hertz's ideas on electromagnetic theory [16] originally stemmed from Helmholtz, who started out as a follower of Weber rather than Maxwell. In the long run, however, Helmholtz's theories were not a great success [17, p. 113] and so Hertz came to embrace Maxwell's equations [18, p. 20]:

> "I have rather been guided by Helmholtz's work [but] the physical basis of Helmholtz's theory disappears as soon as action at distance is disregarded. I therefore endeavored to form for myself… the necessary physical conceptions, starting from Maxwell's equations but otherwise simplifying [his] theory as far as possible"

As to his own contribution in this respect versus that of Heaviside, he reveals in his 1890 paper [19]:

> "I have been led to endeavor for some time past to sift Maxwell's formulae and to separate their essential significance from the particular form in which they first happened to appear… Mr. Oliver Heaviside has been working in the same direction ever since 1885…and the simplest form which [his] equations thereby obtain is essentially the same as that at which I arrive. In this respect, Mr. Heaviside has the priority."

Like Heaviside, his aim was to achieve the clarity that was lacking in Maxwell's exposition. Unlike Heaviside, however, he made no attempt to modernize their mathematical structure; in fact, he took as his mathematical basis the same sort of elementary formulation that Maxwell had first used, for example, X,Y,Z for the components of **E** and L,M,N for **H**. After adjustment to a conventional right-hand co-ordinate system, his general form of the duplex equations appeared as

$$(9a)\ldots\begin{cases}\partial_t B_x = \partial_z E_y - \partial_y E_z \\ \partial_t B_y = \partial_x E_z - \partial_z E_x \\ \partial_t B_z = \partial_y E_z - \partial_x E_y\end{cases} \qquad (9b)\ldots\begin{cases}\partial_t D_x = \partial_y H_z - \partial_z H_y - J_x \\ \partial_t D_y = \partial_z H_x - \partial_x H_z - J_y \\ \partial_t D_z = \partial_x H_y - \partial_y H_z - J_z\end{cases} \tag{5i}$$

As did Heaviside, he made them clearer by removing the potentials; he then presented them in separate forms for free space, isotropic insulators, conductors, *etc*. with the general version above [18, p. 211] tacked on at the end. The divergence equations, however, are far from clear. Hertz's 'electric polarization' $(\mathfrak{X},\mathfrak{Y},\mathfrak{Z})$ is actually our **D**, while his 'magnetic polarization' $(\mathfrak{L},\mathfrak{M},\mathfrak{N})$ is actually our **B**. The electric divergence equations [18, pp. 213-4] are then given as

$$\begin{gathered}\partial_x X + \partial_x Y + \partial_z Z = e_{free} \\ \partial_x \mathfrak{X} + \partial_x \mathfrak{Y} + \partial_z \mathfrak{Z} = e_{true}\end{gathered} \tag{5ii}$$

where (X,Y,Z) corresponds to $\mathbf{E}$ and e is our ρ. He goes on to give an identical prescription for magnetic charge so that in both cases not only do the terms *true* and *free* appear to be effectively reversed compared to the present day usage of *total* and *free* respectively*, but also $\nabla\cdot\mathbf{B}=0$ is treated analogously to $\nabla\cdot\mathbf{D}=0$ as being a *local* rather than a *global* condition.

2.5 Ludwig Boltzmann and August Föppl

Boltzmann and Föppl are better known for their contributions to thermodynamics and structural mechanics respectively, but these are fields in which Maxwell also made significant contributions. Although Boltzmann did work at an electromagnetic theory of his own, he later came to be a proponent of Maxwell's doctrine [20]. The fact that he corresponded with both Heaviside and Hertz could well have helped the dissemination in mainland Europe of Maxwell's theory and of Heaviside's recent progress in elucidating it.

Föppl took part in the correspondence with Boltzmann and Heaviside, [17, p. 113]. Being enthusiastic about the Maxwell-Heaviside theory, his main contribution was the publication of a book in 1894, *Introduction to the Maxwellian Theory of Electricity* [21]. This work is significant because, unlike Heaviside's, it was not a collection of papers, it was a proper book. The treatment of vector analysis and electromagnetics is well organized and clear; crucially, being in German, it must have done much to publicize in mainland Europe the ideas of Maxwell and Heaviside, to both of whom it clearly pays homage [17, p. 113]:

> "The works of [Heaviside] have in general influenced my presentation more than those of any other physicist with the obvious exception of Maxwell himself."

While its influence even spread as far afield as Wilson and Gibbs (see the quotation in Section 2.3 above), Föppl's book and name would surely have been much better known today had an English translation been readily available.

2.6 Other "Maxwellians"

A handful of scientists who actively promoted Maxwell's ideas in the closing decades of the 19th Century came to be known as the "Maxwellians" [22, 23] . Heaviside and Hertz were two, and the others were Oliver Lodge, who made advances in radio technology, and George Fitzgerald, of the Lorentz–Fitzgerald contraction. Although neither Lodge nor Fitzgerald contributed to the actual equations, along with the other Maxwellians they played a significant role in getting them generally accepted. However, Föppl and Boltzmann clearly also contributed to this, as did W. D. Niven, who published Maxwell's collected works [24]

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* Some authors do use the term 'real', meaning 'free', but Hertz also used the term 'free' and in an entirely different context. See Hertz's own description [18, pp. 214]. Föppl followed Hertz's definitions in this respect.

and edited and completed the second edition of the *Treatise* (1881), and J. J. Thomson, discover of cathode rays, who revised it for the third and final edition (1891).

2.7 Hendrik Lorentz

In 1892, Lorentz published an article [25] in which he presented Maxwell's theory along lines that closely followed Hertz's approach. He was also aware of Heaviside's work (perhaps through Föppl) and introduced vector notation, using it side by side with the original Maxwell-Hertzian longhand format, for example, he stated [25, ch. 1, p. 11, §7]

> "The current …will be represented by $\mathbf{C}$, with *u*, *v*, *w* as short-form for $\mathsf{C}_x, \mathsf{C}_y, \mathsf{C}_z$ … The magnetic force and its components [will be represented] by $\mathbf{H}$, α, β, γ, and the magnetic induction and its components by $\mathbf{B}$, a, b, c."

By the time of his 1902 paper [26], however, Lorentz had more fully taken to vector analysis, but using a notation that was somewhat different to that of any of the aforementioned adherents. His goal was to develop the fundamental equations of electromagnetics via his electron theory, and in so doing he gave us Maxwell's equations as a *fundamental set* for what appears to be the first time

$$
\begin{aligned}
&(\mathrm{I})\ldots && \operatorname{div}\mathbf{d} = \rho \\
&(\mathrm{V})\ldots && \operatorname{div}\mathbf{h} = 0 \\
&(\mathrm{IV})\ldots && -c^2 \operatorname{rot}\mathbf{d} = \partial_t \mathbf{h} \\
&(\mathrm{III})\ldots && \operatorname{rot}\mathbf{h} = \partial_t \mathbf{d} + \rho\mathbf{v} \\
&(\mathrm{VI})\ldots && \mathbf{f} = \mu_0\left(c^2\mathbf{d} + \mathbf{v}\times\mathbf{h}\right)
\end{aligned}
\tag{6}
$$

In these equations *rot* means the same thing as *curl* (Maxwell originated both *rotation* and the more familiar term *curl*). In addition, Lorentz deliberately uses **d** and **h** rather than the expected **D** and **H** because he is referring not to these *macroscopic* fields, but to the *microscopic* fields $\mathbf{d} = \varepsilon_0\mathbf{E}$ and $\mathbf{h} = \mathbf{B}/\mu_0$, that is to say, the fields we would encounter within matter by accounting for every iota of static and moving charge, here represented by ρ and $\rho\mathbf{v}$ respectively*. These were to be his underpinning of Maxwell's essentially *macroscopic* equations, in which the usual macroscopic fields are to be seen as arising from the microscopic ones by a process of spatial averaging. With these, Lorentz also includes equation (VI), which expresses the force acting on a unit charge, as being fundamental; it is of course equivalent to the present day form of the Lorentz force, Equation (1)ii, that actually defines what we mean by **E** and **B**.

From this microscopic basis, Lorentz then goes on to recover Equations (1) in the form

* In reality, separate charge densities are required for the static and moving charges.

$$\begin{aligned}
&(\mathrm{I}')\ldots && \operatorname{div}\mathbf{D} = \rho \\
&(\mathrm{V}')\ldots && \operatorname{div}\mathbf{B} = 0 \\
&(\mathrm{IV}')\ldots && \operatorname{rot}\mathbf{E} = -\partial_t\mathbf{B} \\
&(\mathrm{III}')\ldots && \operatorname{rot}\mathbf{H} = \mathbf{J} \qquad \text{where } \mathbf{J} = \partial_t\mathbf{D} + \mathbf{J}^{cond} + \mathbf{C} + \mathbf{R}
\end{aligned} \tag{7}$$

The only real difference is that the free current density $\mathbf{J}^{free}$ is expressed as the sum of three terms: a conduction current $\mathbf{J}^{cond}$, a convection current* **C**, and a Röntgen current **R**. The last two terms, which are are related to the motions of the body of a medium, or within it, are now rarely split out in this way. Lorentz also gave us the full constitutive equation for dielectrics, $\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P}$, which goes beyond Maxwell's concept of displacement by distinguishing the *in vacuo* displacement $\varepsilon_0\mathbf{E}$ from the *physical* electrical polarization **P**. Although he used the term 'measurable' rather than 'free', he nevertheless remedied Hertz's contrary definition of 'free' sources.

Strangely, the name Maxwell appears nowhere in this article, and the same is true of a later one in which Lorentz refers to equations (1)i and (1)ii as "the fundamental equations of the theory of electrons", giving only "M. E." as the source [27, §3]. However, this abbreviation did not mean 'Maxwell's equations', rather it indicated a previous article of his in *Mathematischen Encyklopädie* [28]. Perhaps he now felt that his *electron theory* had superseded Maxwell's original phenomenological theory. Nevertheless, the said article in "M. E." frequently refers to Maxwell, and by embracing the vector analysis of Heaviside and Gibbs, carefully reconstructing the Maxwellian groundwork, tackling the microscopic model and focusing on the essential equations, all of which he published in detail, Lorentz's contribution was highly significant.

3 OTHER FORMS

If the advances made by Heaviside and Lorentz led to Equations (1)–(1)ii in more or less their present day forms, what changes have taken place since? By and large, this version continues to be widely used because the vector analysis of Heaviside and Gibbs is still a cornerstone of the mathematical framework of engineering, physics and applied mathematics. But in advanced subjects like mathematical physics, there is a need for alternative frameworks that are capable of dealing with such things as special and general relativity, the efficient computation of fields, and new concepts in general. As a result, we are likely to come across many other forms of the equations throughout the literature from the early 1900's to the present day. Maxwell himself would have agreed that any mathematical innovation that would reveal more about the true nature of the laws of electrodynamics, or at

* In Lorentz's use of the term, a convection current was not the same as an eddy current

least make them clearer and easier to understand, would be worth looking at. Indeed, he himself had taken that step in embracing quaternions – the rest of the scientific world, on the other hand, seemed not to be quite ready for them at the time. We now trace the key recent developments.

3.1 Hermann Minkowski and Albert Einstein

In the early 20th Century, special relativity was brought to bear on electromagnetic theory with a startling explanation of the negative results of the Michelson-Morley experiment: there is no material ether, the speed of light *in vacuo* is a universal constant, and we live in a four dimensional world. It was Einstein who made the major breakthrough with the theory of special relativity, and in particular his revelation that the observed magnetic field of a moving charge originates from a *purely electric* phenomenon, the Coulomb field of the charge in its own rest frame [15]. More generally, he showed that the laws of electrodynamics were *covariant* [29, 30][*], that is to say, Maxwell's equations apply in any reference frame, in spite of the observed fields, sources and coordinates all appearing different, for example:

$$\text{(unprimed frame)}\quad \nabla\times\mathbf{E}=-\partial_t\mathbf{B} \quad\leftrightarrow\quad \nabla'\times\mathbf{E}'=-\partial_{t'}\mathbf{B}' \quad \text{(primed frame)}$$

It was Minkowski, however, who brought the concept of spacetime to bear on Lorentz's rendition of Maxwell's equations in his 1908 paper [31]. He showed that not only could time and position be combined as one four-vector $[x]$, charge and current (including intrinsic magnetic current) could be combined to form another 4-vector, $[J]$. In a similar vein, the components of **E** and **B** may be combined to form a 4-matrix $[F]$, specifically

$$[x]=\begin{bmatrix} x \\ y \\ z \\ jct \end{bmatrix};\quad [\partial]=\begin{bmatrix} \partial_x \\ \partial_y \\ \partial_z \\ \partial_{jct} \end{bmatrix};\quad [J]=\begin{bmatrix} Z_0J_x \\ Z_0J_x \\ Z_0J_x \\ j\rho/\varepsilon_0 \end{bmatrix};\quad [F]=\begin{bmatrix} 0 & cB_z & -cB_y & -jE_x \\ -cB_z & 0 & cB_x & -jE_y \\ cB_y & -cB_x & 0 & -jE_z \\ jE_x & jE_y & jE_z & 0 \end{bmatrix} \qquad \text{(8)i}$$

where c is the speed of light, and Z_0 is the characteristic impedance, in free space. As a result, Maxwell's equations *in free space* could be expressed as two sets of four simultaneous linear equations

[*] The subject of Einstein's priority over Lorentz and Poincaré in this matter is discussed in the given references.

$$(A)\ldots\quad\begin{array}{cccccc} & \partial_2 F_{12} & +\partial_3 F_{13} & +\partial_4 F_{14} & = & J_1 \\ \partial_1 F_{21} & & +\partial_3 F_{23} & +\partial_4 F_{24} & = & J_2 \\ \partial_1 F_{31} & +\partial_2 F_{32} & & +\partial_4 F_{34} & = & J_3 \\ \partial_1 F_{41} & +\partial_2 F_{42} & +\partial_3 F_{43} & & = & J_4 \end{array}$$

(8)ii

$$(B)\ldots\quad\begin{array}{cccccc} & \partial_2 F_{34} & +\partial_3 F_{42} & +\partial_4 F_{23} & = & 0 \\ \partial_1 F_{43} & & +\partial_3 F_{14} & +\partial_4 F_{31} & = & 0 \\ \partial_1 F_{24} & +\partial_2 F_{41} & & +\partial_4 F_{12} & = & 0 \\ \partial_1 F_{32} & +\partial_2 F_{13} & +\partial_3 F_{21} & & = & 0 \end{array}$$

For example, $\partial_1 F_{41} + \partial_2 F_{42} + \partial_3 F_{43} = J_4$ equates to $\partial_x jE_x + \partial_y jE_y + \partial_z jE_z = j\rho/\varepsilon_0$, that is to say, $\nabla \cdot \mathbf{E} = \rho/\varepsilon_0$. In the case of general media, he obtained a very similar result employing D_k and H_k in the four inhomogeneous equations rather than the E_k and B_k; $[J]$ is then restricted to free sources and the constant factors ε_0 and Z_0 disappear.

By 1916, Einstein had presented his landmark paper on general relativity [32] wherein the treatment of gravitation depended heavily on the use of the tensor formalism that had been quite recently developed by Ricci and Levi-Civita [33]. In that paper he applied their formalism to the free-space Maxwell's equations* and deduced their most succinct form thus far

$$\begin{gathered} \partial_\nu F^{\mu\nu} = J^\mu \\ \partial_\rho F_{\sigma\tau} + \partial_\sigma F_{\tau\rho} + \partial_\tau F_{\rho\sigma} = 0 \end{gathered} \tag{9}$$

For present purposes, both $F^{\mu\nu}$ and $F_{\mu\nu}$ may be read as Minkowski's $F_{\mu\nu}$, and similarly for J^μ. Summation over the repeated index ν is implied in the first equation, whereas in the second ρ, σ and τ are any three of the four indices, leading to a different linear equation for each choice of the single index that is omitted. While this tensor rendition simply appears to restate Minkowski's version in a more elegant and formal way, a key advantage is that the covariance of Maxwell's equations is readily demonstrated [1, §§11.8-9, pp. 374-80].

By dropping the rigorous tensor formalism and turning to ordinary matrix algebra, the equations of Minkowski and Einstein may be written in a form that is more transparent than the former, yet more compact than the latter. With the inclusion of the auxiliary fields **D** and **H**, this leads to [34, §1.21]

$$[\partial]^{\mathrm{T}}[F] = 0; \qquad [\partial]^{\mathrm{T}}[G] = [J] \tag{10)i}$$

* In referring to Maxwell's equations, Einstein had by now dropped his previous inclusion of Hertz's name.

Here $[\partial]^{\mathrm{T}}$ is the transpose of $[\partial]$, *i.e.* the row vector $[\partial_x, \partial_y, \partial_z, \partial_t]$, while $[F]$, $[G]$ and $[J]$ are

$$[F]=\begin{bmatrix} 0 & -E_z & E_y & -jB_x \\ E_z & 0 & -E_x & -jB_y \\ -E_y & E_x & 0 & -jB_z \\ jB_x & jB_y & jB_z & 0 \end{bmatrix}; \quad [G]=\begin{bmatrix} 0 & -H_z & H_y & jD_x \\ H_z & 0 & -H_x & jD_y \\ -H_y & H_x & 0 & jD_z \\ -jD_x & -jD_y & -jD_z & 0 \end{bmatrix}; \quad [J]=\begin{bmatrix} J_x \\ J_y \\ J_x \\ j\rho \end{bmatrix} \qquad (10)\text{ii}$$

in which the J_k and ρ are now restricted to the *free* sources.

3.2 Elie Cartan

Still early in the 20th Century, Cartan was a leading light in the development of differential forms [35], which were to become a key tool of mathematical physics and advanced electromagnetic theory [36; 37]. A close link exists between differential forms and the integral form of Maxwell's equations but, before we expand on that premise, we give a simplified and very basic sketch of how they work.

Starting from the basic notion of a differential, *e.g.* $df(x,y,z) = \partial_x f\,dx + \partial_y f\,dy + \partial_z f\,dz$, the similar looking 1-form $f = f_x dx + f_x dy + f_z dz$ is quite distinct. In fact, in a generalized way it corresponds to an ordinary vector to the extent that the infinitesimal scalar quantities dx, dy and dz may also be treated as independent symbols, like $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$ but with entirely different connotations. Provided the meaning of $\leftrightarrow$ is limited to this sort of correspondence, we may therefore write

$$f = f_x dx + f_y dy + f_z dz \quad \leftrightarrow \quad \mathbf{f} = f_x \hat{\mathbf{x}} + f_y \hat{\mathbf{y}} + f_z \hat{\mathbf{z}}$$

but note that while unit vectors are dimensionless, the differentials dx, dy, dz are not. Extending the idea, a 2-form corresponds to an axial vector (or to a bivector) thus

$$U = U_x dydz + U_y dzdx + U_z dxdy \quad \leftrightarrow \quad \boldsymbol{U} = U_x \boldsymbol{x} + U_y \boldsymbol{y} + U_z \boldsymbol{z}$$

where $\boldsymbol{x} = \hat{\mathbf{y}} \times \hat{\mathbf{z}}$, $\boldsymbol{y} = \hat{\mathbf{z}} \times \hat{\mathbf{x}}$ and $\boldsymbol{z} = \hat{\mathbf{x}} \times \hat{\mathbf{y}}$. While fg, the direct product of two 1-forms f and g, will clearly include a 2-form in the result, it is their *exterior* product, denoted by $f \wedge g$, that produces *exclusively* a 2-form. Defined as being antisymmetric, $f \wedge g = -g \wedge f$, so that the exterior product is related to the direct product through $f \wedge g = (fg - gf)/2$. In particular, if du and $d\upsilon$ are any two of the differentials dx, dy and dz, then $du \wedge d\upsilon = -d\upsilon \wedge du$, whereas $du \wedge du = d\upsilon \wedge d\upsilon = 0$. It is customary, however, to drop the $\wedge$ in these products and simply write $dxdy$ and $dydz$, *etc*.

Along similar lines, the exterior product of a 1-form with a 2-form yields a 3-form, but in this case the product is *symmetric*, as may be inferred from the example:

$$dx \wedge (dy \wedge dz) = dx \wedge dy \wedge dz = -dy \wedge dx \wedge dz = dy \wedge dz \wedge dx = (dy \wedge dz) \wedge dx$$

The general commutation rule is therefore that the exterior product is symmetric when the sum of the degrees of the forms involved is odd, and antisymmetric when it is even.

A key 1-form is the exterior derivative d which takes the form $\partial_x dx + \partial_y dy + \partial_z dz$. By way of example, the exterior derivative of U is $d \wedge U$, commonly written as dU, so that

$$dU \equiv d \wedge U \equiv \left(\partial_x dx + \partial_y dy + \partial_z dz\right) \wedge U$$

For example, $d \wedge x = \left(\partial_x dx + \partial_y dy + \partial_z dz\right) \wedge x = dx$ and $d \wedge (xy) = ydx + xdy$. It is therefore clear that the exterior derivative of a scalar function is a prescription for its differential. Applying it to a 1-form, however, we find

$$\begin{aligned} dE &= \left(\partial_x dx + \partial_y dy + \partial_z dz\right) \wedge \left(E_x dx + E_y dy + E_z dz\right) \\ &= \left(\partial_x E_y - \partial_y E_x\right) dxdy + \left(\partial_y E_z - \partial_z E_y\right) dydz + \left(\partial_z E_x - \partial_x E_z\right) dzdx \end{aligned}$$

$$\Leftrightarrow \quad dE \leftrightarrow \nabla \times \mathbf{E}$$

That is to say, in differential forms, dE takes the place of $\nabla \times \mathbf{E}$. In the case of a 2-form, however, by applying the commutation rules we find

$$\begin{aligned} dD &= \left(\partial_x dx + \partial_y dy + \partial_z dz\right) \wedge \left(D_x dydz + D_y dzdx + D_z dxdy\right) \\ &= \left(\partial_x D_x + \partial_y D_y + \partial_z D_z\right) dxdydz \end{aligned}$$

$$\Leftrightarrow \quad dD \leftrightarrow \nabla \cdot \mathbf{D}$$

In contrast to the case with a 1-form such as E, dD takes the place of $\nabla \cdot \mathbf{D}$ rather than $\nabla \times \mathbf{D}$. Noting that the differential 3D volume element *dxdydz* appears in the result, this is an example of a 3-form, a class that corresponds to a scalar volume density. In 3D we are then left with one other sort of form, the 0-form, a form of scalar that is free from any association with a volume density.

Conventionally, the ubiquitous electromagnetic quantities and source densities are represented by different degrees of forms as follows

0-forms	$q, \quad c, \quad \phi$	$(du)^0 \equiv 1$	1-forms	$E, \quad H, \quad A$	dx, dy, dz
2-forms	$D, \quad B, \quad J$	$dydz, dzdx, dxdy$	3-forms	ρ	$dxdydz$

In each case the associated differential elements are shown in the column to the right of the given symbol. The physical significance of these becomes clearer when we note that $E_x dx$ is the decrease in the potential ϕ of a unit charge when, in vector terms, it is moved through an electric field $\mathbf{E}$ by infinitesimal displacement $dx\hat{\mathbf{x}}$. We also note $E = -d\phi \leftrightarrow \mathbf{E} = -\nabla\phi$.

Similarly, the 2-form quantities may be associated with a flux, so that, for example, $J_z dxdy$ is the total current that flows through the orientated element of area $dxdy$, *i.e.*, the area $dxdy$ is in the *xy* plane such that a positive flow is along $\hat{\mathbf{x}} \times \hat{\mathbf{y}} = \mathbf{z}$. Finally, the 3-form is fairly self-evident inasmuch as $\rho dxdydz$ is the total charge q contained within the volume element $dxdydz$.

Following these preliminaries, it should be clear that the result of applying the operator d depends on the degree of the form that it acts on, so that ∇, $\nabla \times$ and $\nabla \cdot$ are *all* replaced by the single operator d (meaning $d \wedge$) on its own. The expression of Maxwell's equations in terms of differential forms is therefore very straightforward, for we can use this rule to transcribe Equations (1) into

$$\begin{aligned} dD &= \rho^{free} \\ dB &= 0 \\ dE &= -\partial_t B \\ dH &= J^{free} + \partial_t D \end{aligned} \tag{11}$$

These are then a direct source for the integral equations which are exactly mirrored by

$$\begin{aligned} \int_{\partial V} dD &= \int_V \rho^{free} \\ \int_{\partial V} dB &= \int_V 0 = 0 \\ \int_{\partial A} dE &= -\partial_t \int_A B \\ \int_{\partial A} dH &= \int_A \left(J^{free} + \partial_t D \right) \end{aligned} \tag{12}$$

It has been necessary only to write integral signs on both sides of Equation (11), with the degree of the form telling us what sort of integral is involved: line, surface or volume. The integrals on the left-hand side are taken over the closed boundary of the volume or area associated with the integrals on the right-hand side, that is to say, ∂A is the closed path taken around the outside of the area *A* and ∂V is the surface enclosing the volume *V*. Because the integrands are differential forms, not only are the requisite differentials for the integration already in place, they also provide the orientation of the paths and surfaces, *e.g*. *dx* is along $\hat{\mathbf{x}}$ and the normal to $dxdy$ is $\mathbf{z}$.

Equations (11) may be put in spacetime form by making the following extensions [37; 38]

$$d \equiv \left(\partial_x dx + \partial_y dy + \partial_z dz + \partial_t dt \right)$$

As in Minkowski's matrix, the single 2-form *F* now represents the complete electromagnetic field. Likewise, the auxiliary fields combine into a separate 2-form, *G*, while the free current and charge densities combine into a single 3-form *J*. The equations may now be written as

$$dF = 0; \qquad dG = J^{free}$$

$$\text{where} \quad F = E \wedge dt + B; \quad G = -H \wedge dt + D; \quad J = -J^{free} \wedge dt + \rho^{free} \qquad (13)$$

By applying the same simple rules as before and writing d as $\left(\underset{\sim}{d} + \partial_t dt\right)\wedge$, where $\underset{\sim}{d}$ represents the *original* 3D exterior derivative $\partial_x dx + \partial_y dy + \partial_z dz$, this may be decoded in the following manner,

$$\begin{aligned} dF &= \left(\underset{\sim}{d} + dt\partial_t\right) \wedge E \wedge dt + \left(\underset{\sim}{d} + dt\partial_t\right) \wedge B \\ &= \underset{\sim}{d}E \wedge dt \;-\; \partial_t E \wedge dt \wedge dt \;+\; \underset{\sim}{d}B \;+\; \partial_t B \wedge dt \quad ; \\ &= \left(\underset{\sim}{d}E + \partial_t B\right) \wedge dt \;+\; \underset{\sim}{d}B \\ &= \qquad 0 \qquad + \;\; 0 \end{aligned}$$

$$\begin{aligned} dG &= -\left(\underset{\sim}{d} + dt\partial_t\right) \wedge H \wedge dt + \left(\underset{\sim}{d} + dt\partial_t\right) \wedge D \\ &= -\underset{\sim}{d}H \wedge dt \;+\; \partial_t H \wedge dt \wedge dt \;+\; \underset{\sim}{d}D \;+\; \partial_t D \wedge dt \\ &= \left(-\underset{\sim}{d}H + \partial_t D\right) \wedge dt \;+\; \underset{\sim}{d}D \\ &= \qquad -J^{free} \qquad + \;\; \rho^{free} \end{aligned}$$

Given an operator $*$ that, in 3D, converts a 1-form into a 2-form (its dual) and *vice versa*, the substitutions $H = \mu * B$ and $D = \varepsilon * E$ may be made, but this still leaves us with two separate equations.

3.3 David Hestenes

More recently, David Hestenes [39; 40] and others began promoting geometric algebra, reviving the 19th Century work of Hermann Grassmann and William Clifford. This concept provides an equally powerful counterpart to differential forms and embodies some of the useful features of quaternions, for example, inverses. A geometric algebra is a vector space in which the roles of second rank skew-symmetric tensors and differential 2-forms are both replaced by, the bivector, a single entity formed by the direct multiplication of two orthogonal vectors. This is in strict contrast to the quaternions where such a product effectively reverts to a vector. As there is no need to continually refer to some assumed spatial frame such as $\{\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}\}$, geometric algebra provides a versatile coordinate-free approach. This means that rather than effectively being labels for ordered components, symbols such as $\mathbf{E}$ and ∇ actually stand for the vectors themselves. Nevertheless, we may still express these vectors as $\mathbf{E} = E_x\hat{\mathbf{x}} + E_y\hat{\mathbf{y}} + E_z\hat{\mathbf{z}}$ and $\nabla = \hat{\mathbf{x}}\partial_x + \hat{\mathbf{y}}\partial_y + \hat{\mathbf{z}}\partial_z$, or in terms of whatever other basis we may find convenient.

The geometric algebra formalism of multivectors allows for a *graded* hierarchy of entities called *n*-vectors, where a 0-vector is a scalar, a 1-vector is the familiar sort of vector, a 2-vector is a bivector, and so on. It is a key feature that different grades of *n*-vector may be added as well as multiplied, resulting in what is generally a multivector of mixed grade. In order to distinguish them from the usual 3D vectors, we will write general *n*-vectors and multivectors in bold italic, *e.g.* $\boldsymbol{u}, \boldsymbol{v}$ and so on. By the basic rules of geometric multiplication, the product $\boldsymbol{uv}$ resolves into the scalar $\boldsymbol{u}\cdot\boldsymbol{v}$ plus the bivector $\boldsymbol{u}\wedge\boldsymbol{v}$. Note that the wedge symbol $\wedge$ used in differential forms conveys a similar idea. For 3D vectors, $\mathbf{u}\wedge\mathbf{v} = I\mathbf{u}\times\mathbf{v}$ where $I = \hat{\mathbf{x}}\hat{\mathbf{y}}\hat{\mathbf{z}}$ is a 3-vector referred to as the unit pseudoscalar. While I takes a role analogous to the imaginary unit *j*, it has the additional property that when it multiplies a vector it creates a *bivector*. In this fashion, therefore, $\mathbf{uv} = \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\wedge\mathbf{v} = \mathbf{u}\cdot\mathbf{v} + I\mathbf{u}\times\mathbf{v}$, which leads us straight to

$$\begin{aligned}\nabla\mathbf{E} &= \nabla\cdot\mathbf{E} + I\nabla\times\mathbf{E} \\ &= \frac{1}{\varepsilon_0}\rho - \frac{1}{c}\partial_t\left(Ic\mathbf{B}\right)\end{aligned} \quad \text{and} \quad \begin{aligned}\nabla\left(Ic\mathbf{B}\right) &= Ic\nabla\cdot\mathbf{B} + I^2c\nabla\times\mathbf{B} \\ &= -Z_0\mathbf{J} - \frac{1}{c}\partial_t\mathbf{E}\end{aligned} \quad \text{where } Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}$$

Since the addition of different grades is permitted, we find that we may put both of these results together so as to render Maxwell's equations in free space as a single '(3+1)D' equation*

$$\left(\nabla + \partial_t\right)\boldsymbol{F} = \boldsymbol{J} \tag{14}$$

in which the entire electromagnetic field is expressed as the multivector $\boldsymbol{F} = \mathbf{E} + Ic\mathbf{B}$, and likewise the total electromagnetic source density is expressed as the multivector $\boldsymbol{J} = \frac{1}{\varepsilon_0}\rho - Z_0\mathbf{J}$.

We can go on to define an auxiliary electromagnetic field, $\boldsymbol{G} = \frac{1}{\varepsilon_0}\mathbf{D} + IZ_0\mathbf{H}$, the role of which is the macroscopic treatment of physical media. Maxwell's equations then become [41, §5.9]

$$\begin{aligned}\left\langle\left(\nabla + \partial_t\right)\boldsymbol{G}\right\rangle_{0,1} &= \boldsymbol{J}^{\,free} \\ \left\langle\left(\nabla + \partial_t\right)I\boldsymbol{F}\right\rangle_{0,1} &= 0\end{aligned} \tag{15}$$

The subscripts 0,1 on the angled brackets indicate that only the 0- and 1-vector (scalar and vector) grades are to be taken from the result of the enclosed expression ($\langle..\rangle_0$ and $\langle..\rangle_1$ are directly analogous to Maxwell's *S.* and *V.*). There are other ways of writing this, but none in which the electromagnetic field quantities are represented only in terms of the entire fields $\boldsymbol{F}$ and $\boldsymbol{G}$.

In spacetime, however, we get a similar but far more effectual expression of these equations, namely

* Hamilton gave the interpretation (3+1)D for a quaternion such as $t + \mathbf{r}$, the sum of scalar time with a vector position

$$\begin{aligned} \nabla \wedge F &= 0 \\ \nabla \cdot G &= J^{free} \end{aligned} \tag{16}$$

(it should be noted here that the use of bold symbols is usually dropped with these spacetime quantities). Since time is now embodied as a vector in this 4D space, ∇ is now equivalent to $\boldsymbol{x}\partial_x + \boldsymbol{y}\partial_y + \boldsymbol{z}\partial_z - \frac{1}{c}\boldsymbol{t}\partial_t$ in which ***x***, ***y***, ***z*** and ***t*** are the unit basis vectors of some *spacetime* frame. Although J^{free}, *F* and *G* retain similar physical definitions to their 3D mixed-multivector counterparts, J^{free} is now a pure vector and *F* and *G* are both pure bivectors. Electric and magnetic fields are now just the timelike and spacelike* parts of the same bivector field, *i.e.* of *F* or *G* as appropriate, and because Equation (16) works in any inertial frame, it is essentially covariant.

When all is stripped back to a fundamental setting devoid of phenomenological representations for physical media, the auxiliary fields vanish and Maxwell's equations are once again a single equation

$$\nabla F = J \tag{17}$$

in which *J* now comprises *all* sources of charge and current. This equation is simplicity itself. Also covariant, it defines an important class of equation that stems from concepts that seem more abstract than physical. In the more familiar case of complex functions in 2D, $\nabla F = 0$ corresponds to the pair of Cauchy-Riemann conditions $\partial_x F_x - \partial_y F_y = 0$ and $\partial_x F_y + \partial_y F_x = 0$, meaning that *F* must be an analytic function with singularities wherever $J \neq 0$. In spacetime, where there are two extra dimensions, solutions of $\nabla F = 0$ are called meromorphic functions, but otherwise the situation is analogous to 2D. Finally, since all non-null vectors in a geometric algebra have inverses, we may write down a particular solution of Equation (17) in closed form, as simply

$$F = \nabla^{-1} J \tag{18}$$

where ∇^{-1} turns out to be an integral operator with a time-dependent Green's function as its kernel.

4 VARIATIONS

4.1 The Integral Equations

While we normally think of Maxwell's equations as differential equations, as we saw in §3.2 their treatment in terms of differential forms leads directly to a set of integral equations. The integral form is also well known in standard vector analysis [14, pp. 194-6; 34, §1.4; 41, §3.4; 42, ch.1].

* A bivector *U* is timelike if the sign of $UU^{\dagger}$ is of the same as $\boldsymbol{t}^2$, but for a spacelike bivector it is just the opposite.

$$\begin{aligned}
&\int_S \mathbf{D}\cdot d\boldsymbol{A} = q^{free} && (\text{the total charge contained within } V) \\
&\int_S \mathbf{B}\cdot d\boldsymbol{A} = 0 && (\mathbf{B} \text{ is always solenoidal}) \\
&\oint_P \mathbf{E}\cdot d\boldsymbol{l} = -\partial_t \Phi && (-\text{the rate of change of magnetic flux passing through } A) \\
&\oint_P \mathbf{H}\cdot d\boldsymbol{l} = I^{free} + I^{disp} && (\text{the total current passing through } A)
\end{aligned} \tag{19}$$

Here the surface *S* completely encloses a volume *V*, whereas the path *P* completely encloses an area *A*, and $d\boldsymbol{A}$ and $d\boldsymbol{l}$ are orientated infinitesimal elements of *S* and *P* respectively. The magnetic flux Φ passing through *A* is then given by $\int_S \mathbf{B}\cdot d\boldsymbol{A}$ while the displacement *current* I^{disp} is given by $\partial_t \int_S \mathbf{D}\cdot d\boldsymbol{A}$. These equations are perhaps most useful as a way of expressing the physical laws of electromagnetic theory. They have a direct and obvious correspondence to Equations (12) and (11) of differential forms, but in geometric algebra the inner products in the first and last equations change to outer products so to maintain the appropriate grades [41, §3.5].

4.2 Other Ways of Writing the Standard Form

Just how unique is the set of Maxwell's equations as expressed in (1)? There are undoubtedly other ways of writing them which work just as well [2, §4], but Equations (1) are unique to the extent that no constants are required, nor is there any reference to the bound sources that arise from the electric and magnetic polarizations **P** and **M**. However, this situation is quite arbitrary, for the vanishing of the constants is due to the choice of units, and Equations (1)i would allow us to switch from the variables **D**, **B**, **E** and **H** to **P**, **M**, **E** and **B**, or even to **P**, **M**, **E** and **H**, and so on. Maxwell may have set some sort of precedent when he wrote $\mathbf{D} = \varepsilon\mathbf{E}$ but, as an enthusiast of molecular theory, it is very likely that he would have brought in **P** had the microscopic nature of dielectrics been properly understood at that time. He could then have considered **D** to be redundant, or at best auxiliary, for it is **P** that has a tangible meaning directly associated with the state of individual molecules. It would have been just as easy for him to write $\mathbf{P} = \alpha\mathbf{E}$ as it was to write $\mathbf{D} = \varepsilon\mathbf{E}$ (in whichever algebraic form). Therefore, had **P** and **M** been preferred over **D** and **H**, the equations would have come down to us as

$$\begin{aligned}
\nabla\cdot\varepsilon_0\mathbf{E} &= \rho^{free} - \nabla\cdot\mathbf{P} \\
\nabla\cdot\mathbf{B} &= 0 \\
\nabla\times\mathbf{E} + \partial_t\mathbf{B} &= 0 \\
\nabla\times\mathbf{B} - \frac{1}{c^2}\partial_t\mathbf{E} &= \mu_0\left(\mathbf{J}^{free} + \nabla\times\mathbf{M} + \partial_t\mathbf{P}\right)
\end{aligned} \tag{20}$$

While these may seem to be less succinct, no separate constitutive relations are required. Moreover, they are just as workable as Equations (1), which simply tidy things up a bit by replacing two divergences, two curls and two derivatives with just one apiece. In particular, in the context of media with linear polarisabilities, the differences all 'come out in the wash'. In particular, if we put $\mathbf{P} = \alpha\mathbf{E}$ and $\mathbf{M} = \beta\mathbf{B}$ into Equations (20), we get

$$\begin{aligned} \nabla\cdot(\varepsilon_0+\alpha)\mathbf{E} &= \rho^{free} \\ \nabla\cdot\mathbf{B} &= 0 \\ \nabla\times\mathbf{E}+\partial_t\mathbf{B} &= 0 \\ \nabla\times\left(\frac{1}{\mu_0}-\beta\right)\mathbf{B}-\partial_t\left(\frac{1}{\mu_0 c^2}+\alpha\right)\mathbf{E} &= \mathbf{J}^{free} \end{aligned} \tag{21}$$

The usual form is therefore restored by replacing $\varepsilon_0+\alpha$ with ε and $1/\mu_0-\beta$ with $1/\mu$. However, the fact that the 'usual' form is regarded as being 'usual' is clearly a matter of convention rather than necessity. Besides, Equations (20) are manifestly applicable to free space (no substitutions required) for here the polarizations simply vanish, giving

$$\begin{aligned} \nabla\cdot\mathbf{E} &= \rho/\varepsilon_0 \\ \nabla\cdot\mathbf{B} &= 0 \\ \nabla\times\mathbf{E}+\partial_t\mathbf{B} &= 0 \\ \nabla\times\mathbf{B}-\frac{1}{c^2}\partial_t\mathbf{E} &= \mu_0\mathbf{J} \end{aligned} \tag{22}$$

This set is also applicable in a microscopic treatment of media because in this situation bound charge, rather than being eliminated from the total charge, is embodied within it on a molecular basis. An entirely separate observation about the form of Equations (20)–(22) is that all the sources have been placed on the right-hand side whereas all the fields are on the left-hand side, a convention that some people prefer because it clearly separates the dependent and independent variables. This is not entirely pedantic – for one thing it avoids confusing $\varepsilon_0\partial_t\mathbf{E}$, the vacuum contribution to the displacement 'current', with a real current density such as the polarization current $\partial_t\mathbf{P}$ [43].

In the case of the free-space equations, rather than taking **E** and **B** as being the principal fields, some writers advocate keeping **D** and **H** in the inhomogeneous equations as excitations arising from the sources, as distinct from the field strengths **E** and **B** [44]. It has also been commonplace with some writers to take **E** and **H** or even, as Lorentz did, **d** and **h**. This is encouraged by the fact that in free space we can exchange **D** with $\varepsilon_0\mathbf{E}$ and $\mu_0\mathbf{H}$ with **B**. In some instances the justification for doing so is the availability of convenient units, *e.g.* A/m rather than T. In some other systems, however, the units are the same, so that the exchange is all too easy. While such variations are understandable in a historical context, in the present day it should be understood that Equation (1)ii differentiates **E** from **D** and **B** from **H**; consequently, when it comes to the fundamental equations in free space we should use Equations (22) which are in terms of **E** and **B** alone. For example, it is never wrong to say

$\nabla\cdot\mathbf{B}=0$, whereas a statement such as $\nabla\cdot\mathbf{H}=0$ may well become a nonsense if transposed to the case of physical media. The Chu formulation of electrodynamics [42], by adhering to the use of **E** and **H** even in the Lorentz force, causes some difficulties in this respect [45].

4.3 Back to Quaternions

But quaternions have not, as Gibbs and Heaviside may have wished, been altogether consigned to the dust. In particular, there has been recent interest in the application of biquaternions (also known as complex quaternions or octonions) to electromagnetic theory [46]. In the original quaternion theory Maxwell's equations in free space reduce to a pair,

$$\begin{aligned}\nabla\mathbf{E}&=-\frac{1}{\varepsilon_0}\rho-\frac{1}{c}\partial_t c\mathbf{B}\\ \nabla c\mathbf{B}&=Z_0\mathbf{J}+\frac{1}{c}\partial_t\mathbf{E}\end{aligned}\tag{23}$$

This is similar to the result we obtained with geometric algebra, but prior to combining **E** and **B** as a multivector. This suggests that if we were to combine **E** and **B** as the complex quaternion, $\boldsymbol{F}=\mathbf{E}+jc\mathbf{B}$, a further degree of compactness may be possible. Indeed, Maxwell's equations in free space are rendered as a single equation that closely parallels the (3+1)D multivector form they take in a 3D geometric algebra, Equation (14). Simply by adding the first row in Equation (23) to *j* times the second, we obtain

$$\left(\nabla-\frac{j}{c}\partial_t\right)\boldsymbol{F}=-\frac{1}{\varepsilon_0}\rho+jZ_0\mathbf{J}\tag{24}$$

Note that in free space $\left(\nabla-\frac{j}{c}\partial_t\right)\boldsymbol{F}=0$ leads to a wave equation because, instead of the customary vector rule $\nabla^2=\boldsymbol{\nabla}^2$, with quaternions $\nabla^2=\partial_x^2+\partial_y^2+\partial_z^2=-\boldsymbol{\nabla}^2$, which leads to

$$-\left(\nabla+\frac{j}{c}\partial_t\right)\left(\nabla-\frac{j}{c}\partial_t\right)\boldsymbol{F}=\left(\nabla^2-\frac{1}{c^2}\partial_t^2\right)\boldsymbol{F}=0\ .$$

5 CONCLUSION

While we have covered some 24 versions of Maxwell's equations that have come down to us over the years as a result of both evolutionary progress and new directions, these represent only those that are close to the mainstream. The version that we commonly see today in Equations (1) differ from Maxwell's original form, as represented in Equations (2) and Figures 1-3, only in some minor respects, mainly in that he employed

- Less sophisticated mathematics
- The vector and scalar potentials, and
- The constitutive relations $\mathbf{D}=\varepsilon\mathbf{E}$ and $\mathbf{B}=\mu\mathbf{H}$.

Even if it may seem obscure, the quaternion mathematics he used in the *Treatise*, Equations (3), is equivalent to, and no less sophisticated than, today's vector analysis. By this time he had also brought in $\mathbf{B} = \mu_0 \left(\mathbf{H} + \mathbf{M} \right)$, and the fact that he still used potentials amounts only to a different way of expressing things. The changes since then have therefore been due to

- Developments in the mathematical languages of physics
- Advances in knowledge, particularly concerning the nature of electrical charge and the distinction between polarization and electric displacement
- Refinement of his equations into a basic set comprising Equations (1)–(1)ii, the first group of which we now regard as being *Maxwell's equations*.

The direct path of development of the equations was

- 1862, Maxwell's first formulation of his equations based on a molecular vortex model
- 1864, his new field-based formulation of them for the *Dynamical Theory*
- 1873, his subsequent introduction of vectors and quaternions in the *Treatise*
- 1885, Heaviside disposing of quaternions and potentials and using his vector analysis to re-formulate Maxwell's quaternionic equations
- 1902, Lorentz setting down the basic microscopic equations and then using his electron theory *to derive* the usual set of four macroscopic equations.

Others contributed, notably

- Hamilton, who discovered quaternions, and Tait who promoted them
- Gibbs, who independently developed vector analysis, and Wilson, who later published it.
- The Maxwellians, who, along with Boltzmann and Föppl, championed and disseminated Maxwell's work
- Niven and Thomson, who edited the second and third editions of the treatise and did much to clarify it and amend errors
- Hertz, who produced a clarified version the equations and validated the revolutionary prediction of Maxwell's theory – electromagnetic waves that travel with the speed of light.

As to the other and more recent ways of writing Maxwell's equations, it is clear that special relativity has made a major impact*. But contrary to the situation with Newtonian mechanics, this has not meant that the existing equations have had to be either rewritten or treated as approximations for, quite remarkably, they still stand as they were. Rather, it has provided a whole new approach that has led to a better understanding of the foundational principles of

* By virtue of the correspondence principle, Maxwell's equations are also compatible with elementary quantum mechanics, in which a key difference is that the electromagnetic field becomes quantized in the form of photons that exhibit both particle-like and wave-like characters. This discovery resolved the dichotomy between the wave theory of light, advocated by both Robert Hooke and Christiaan Huygens, and the particle theory advanced by Sir Isaac Newton (Hooke's arch rival).

electrodynamics. It is also clear that several different mathematical structures may be used to express the same physical laws: vectors, tensors, quaternions, differential forms, geometric algebra and biquaternions. However, when it comes to seeking the most succinct way of encoding the fundamental equations, it is perhaps unsurprising that there seem to be strong parallels between the available structures. While there can be no single mathematical platform to suit all users and all purposes alike, such parallels tend to suggest that these different formalisms are simply alternative ways of invoking the same underlying physical principles and relationships. If, however, we were to consider brevity worthy of special merit, James Clerk Maxwell would surely have been impressed by the free space Equation (17), $\nabla F = J$, together with the Lorentz force cast as $f = qv \cdot F$.

Ultimately, the form taken by the equations is clearly far less important than their underlying meaning, the essential core of which, at least, must be independent of how the equations themselves are expressed. Therefore, in spite of all the variations, modifications and developments, we must give Maxwell himself the credit for successfully gathering together and setting down the foundations of electromagnetics in terms of equations, albeit in a now unfamiliar form. In so doing, his signal achievement was the establishment of the first viable theory of all electromagnetics as a *field theory*. When early adopters such as the Maxwellians began communicating and clarifying the theory, it rapidly gained supremacy over ideas of action at a distance, and by the closing decade of the 19th century it was all but universally accepted.

6 ADDITIONAL SOURCES OF INFORMATION

Histories concerning the development of electromagnetics and Maxwell's equations,

- [22-24, 47-61]

Information about Maxwell's equations in various forms, including relativistic,

- [38, 62-66]

Vector analysis, differential forms and geometric algebra,

- [62, 67-72]

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